On the Influence of Vapour Pressure on Refraction.
By Dr. L. de Ball.

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The theory of refraction which M. Radau develops in his excellent memoir, "Essai sur les Réfractions astronomiques,"* concerns also the influence of the pressure of the vapours present in the atmosphere.† Although this memoir appeared quite sixteen years ago, refraction, even to this day, is almost without exception computed without any consideration to the dampness of the air. The consequence of this neglect, however, is that the zenith-distances, thus reduced, contain systematic errors of annual period. Since, so far as I know, attention has not yet been called to this point, a brief dissertation on the real state of the matter may not be quite out of place.

Let ρ be the density of the air at the place of observation, c a constant, and put

$$(1) \qquad \frac{c\rho}{1+2c\rho}=a$$

then the refraction for the apparent zenith-distances $z \leqq 75°$ may be represented by the series

$$(2) \qquad \text{Refraction} = \frac{a}{\sin 1''}(a_0 \operatorname{tang} z - a_1 \operatorname{tang}^3 z + a_2 \operatorname{tang}^5 z - a_3 \operatorname{tang}^7 z)$$

The value of the coefficient a varies with the density of the air. If the density of the air have the value ρ_0, and we write

$$(3) \qquad \frac{c\rho_0}{1+2c\rho_0}=a_0$$

then the result is at once

$$(4) \qquad a=\frac{\rho}{\rho_0}\;\frac{a_0}{1-2a_0\left(1-\frac{\rho}{\rho_0}\right)}$$

If, therefore, the real value for any density of the air be known, the value of a for each density can be computed; and when a is known and the temperature besides, the sum of the above series may also be ascertained. The density is found from the readings of the barometer, of the interior and exterior thermometer, in connexion with the vapour-pressure, with the height above sea-level of the place of observation and its geographical latitude;

* *Annales de l'Observatoire de Paris*, t. xix.

† Older theories, in which the influence of the dampness in the air is at least touched upon, are found mentioned in C. Bruhns' *Die astronomische Strahlenbrechung*. C. Bruhns, however, says in conclusion (p. 173): "Rightly, therefore, have Bessel, . . . paid no further attention to the damp air."

and taking for the unit of density the density of dry air at 0° Celsius, under pressure of one atmosphere (column of mercury 760 mm.), if B is the reading of the barometer (millimetres), τ the reading of the interior thermometer (Celsius), h the height of the point of observation above sea-level, expressed in metres, and ϕ its geographical latitude, then the air-pressure, p, follows from the formula

$$(5)\quad p=(1-0{\cdot}000000196h-0{\cdot}00265\cos 2\phi)\frac{\mathrm{B}}{760}(1-0{\cdot}000162\tau)$$

The air constantly contains a certain amount of vapour. According to the researches of Dalton, the pressure of a mixture of air and vapour, at the temperature t, is equal to the sum of their separate pressures at the same temperature. Therefore, if p be the pressure of the damp air (which can be obtained from the reading of the barometer by the foregoing formula) and if π be the pressure of the vapour contained in the air, then $p-\pi$ is the pressure which the dry air alone exerts. According to the Gay-Lussac–Mariotte law, however, the quotient for dry air

$$\frac{\text{Pressure}}{\text{Density}\times(1+at)}$$

where $a=0{\cdot}003663$ (Regnault) is the expansion coefficient of dry air, is a constant. As for $p=1$ and $t=0°$ C. the density is to be $=1$, so must the above-mentioned constant be equal to 1; hence the equation is for dry air at the temperature t

$$(6)\qquad \text{Density}=\frac{\text{Pressure}}{1+at}$$

If, therefore, we indicate the density of the dry air of the temperature t, and under the pressure $p-\pi$ by ρ_1, we have

$$\rho_1=\frac{p-\pi}{1+at}$$

If the vapour of the temperature t, whose pressure is indicated by π, be replaced by dry air of the temperature t, which exerts the same pressure π, the density of this air, according to equation (6), would be equal $\dfrac{\pi}{1+at}$. Experiment, however, shows that the weight of a volume of vapour is only 0·622 of the weight of an equal volume of dry air at the same temperature and pressure, and therefore, for like temperature and like pressure, the density of the vapour is also 0·622 of the density of the dry air; and we have for the density ρ_2 of the vapour

$$\rho_2=0{\cdot}622\frac{\pi}{1+at}$$

Now the density of the damp air is equal to the sum $\rho_1+\rho_2$, and writing the previously ascertained values of these we have

$$\rho=\frac{p-0{\cdot}378\pi}{1+at}$$

or if 0·378 be replaced by $\frac{3}{8}$

$$(7) \qquad \rho=p\frac{1-\frac{3}{8}\frac{\pi}{p}}{1+at}$$

If the vapour pressure be expressed in millimetres by the height of a mercury column, whose pressure is equal to that of the humidity contained in the air, and if the height of this column of mercury be likewise indicated by π, then it follows from equation (5), and from the corresponding equation for the vapour pressure, that in equation (7) $\frac{\pi}{p}$ may be replaced by $\frac{\pi}{B}$; and from the same equations (5) and (7), with the above assigned value of a, we obtain

$$\rho=\frac{B}{760}\frac{1-0{\cdot}000162\tau}{1+0{\cdot}003663t}(1-0{\cdot}000000196h-0{\cdot}00265\cos 2\phi)\left(1-\frac{3}{8}\frac{\pi}{B}\right)$$

or

$$(8) \quad \rho=\frac{B}{760}\frac{1-0{\cdot}000162t}{1+0{\cdot}003663t}[1-0{\cdot}000000196h-0{\cdot}00265\cos 2\phi+0{\cdot}000162(t-\tau)]\left(1-\frac{3}{8}\frac{\pi}{B}\right)$$

For $B=760^{mm}$, $\tau=0°$, $h=0$, $\phi=45°$, $\pi=6^{mm}$, therefore for the density

$$(9) \qquad \rho_0=1-\frac{3}{8}\frac{6}{760}$$

Professor Bauschinger gives for value of the refraction constant $\alpha_0=60''{\cdot}15\sin 1''$. We have

$$\frac{1-\frac{3}{8}\frac{\pi}{B}}{1-\frac{3}{8}\frac{6}{760}}=1+\frac{3}{8B}\left(6\frac{B}{760}-\pi\right)$$

If we put

$$(10) \quad B[1-0{\cdot}000000196h-0{\cdot}00265\cos 2\phi+0{\cdot}000162(t-\tau)]+\frac{3}{8}\left(6\frac{B}{760}-\pi\right)=\beta$$

we find from the equations (8) and (9)

$$(11) \qquad \frac{\rho}{\rho_0}=\frac{\beta}{760}\frac{1-0{\cdot}000162t}{1+0{\cdot}003663t}$$

To form a serviceable table for the calculation of refraction we can now compute the quotient $\frac{\rho}{\rho_0}$ for a series of equidistant values of β and t, and then from (4) [by the help of $\frac{\rho}{\rho_0}$ and $\alpha_0 = 60''\cdot 15 \sin 1''$ corresponding to ρ_0] we have the values of α; and when these are found the refractions corresponding to the adopted series of values of β and t can be calculated by (2). In order, therefore, to draw from the table thus obtained the refraction corresponding to the observed values of B, t, τ and π, there is the correction

$$\tfrac{3}{8}\left(6\frac{\mathrm{B}}{760}-\pi\right)$$

to be applied to the indicated barometrical height B, besides the corrections dependent on h, ϕ, and $t-\tau$. The influence which this class of corrections has on refraction may easily be calculated by the aid of M. Radau's tables. The table marked I. by M. Radau indicates the mean refraction; Table II. gives, together with the arguments, zenith distance, and temperature, the variation of the mean refraction for 1° C., this variation—supposing that it is always taken as negative—is to be multiplied by t and added to the mean refraction. From Table IV., with the argument, mean refraction + correction for temperature, we can at length derive the variation of the refraction which corresponds to the change of 1^{mm} of the barometer. If this variation be multiplied by $\tfrac{3}{8}\left(6\frac{\mathrm{B}}{760}-\pi\right)$ the required influence of the vapour pressure on refraction is the result.

I now give a list of the monthly means of π and t, which has been compiled from the observations of Professor Bauschinger in Munich (1891–93) and of Dr. Grossmann * in Vienna-Ottakring (1896–98):

	Munich.		Ottakring.	
	π	t	π	t
	mm.	°	mm.	°
January	1·8	−12	...	
February	4·1	+ 1	3·9	+ 1
March	4·1	+ 3	4·2	+ 6
April	4·3	+ 7	5·7	+ 9
May	8·2	+14	8·9	+14
June	9·6	+14	10·4	+19
July	10·5	+17	...	
August	9·1	+14	12·7	+19
September	9·5	+12	10·4	+16
October	7·0	+ 7	7·1	+ 9
November	5·0	+ 1	4·0	+ 2
December	3·7	− 2	3·9	0

* "Beobachtungen am Repsoldschen Meridiankreise der von Kuffnerschen Sternwarte in den Jahren 1896–98," *Sitzungsberichte der königl. Sächsischen Gesellschaft der Wissenschaften*, Band xxvii. No. 1.

If we write $\frac{3}{8}(6-\pi)$ instead of $\frac{3}{8}\left(6\,\frac{B}{760}-\pi\right)$, which, however, is not strictly correct, we get for the influence of vapour pressure on refraction, at the zenith distances 55°, 65°, 70°, 75°, the following values :

	Munich.				Vienna-Ottakring.			
	55°.	65°.	70°.	75°.	55°.	65°.	70°.	75°.
	″	″	″	″	″	″	″	″
January	+0·19	+0·29	+0·37	+0·50	...	...	...	...
February	+0·08	+0·12	+0·15	+0·20	+0·09	+0·14	+0·18	+0·23
March	+0·08	+0·12	+0·15	+0·20	+0·08	+0·12	+0·15	+0·20
April	+0·07	+0·10	+0·13	+0·17	+0·01	+0·02	+0·02	+0·03
May	−0·09	−0·13	−0·16	−0·22	−0·11	−0·18	−0·23	−0·31
June	−0·14	−0·21	−0·26	−0·36	−0·18	−0·26	−0·32	−0·43
July	−0·18	−0·27	−0·34	−0·46	...	...	...	...
August	−0·13	−0·19	−0·24	−0·34	−0·27	−0·40	−0·50	−0·67
September	−0·14	−0·21	−0·27	−0·36	−0·18	−0·26	−0·32	−0·45
October	−0·04	−0·07	−0·08	−0·11	−0·04	−0·06	−0·08	−0·11
November	+0·04	+0·07	+0·09	+0·12	+0·08	+0·12	+0·15	+0·20
December	+0·10	+0·15	+0·20	+0·26	+0·09	+0·14	+0·18	+0·23

The errors arising from the neglect of the vapour pressure are noticeable even in the mean zenith distances; they have moreover for the two specified series of observations from November to April the opposite signs to those from May to October. The zenith distances, without the corrections for vapour pressure, are too small in winter and too great in summer. The previously calculated corrections of refraction on account of vapour pressure can be represented by $f+x\sin\odot+y\cos\odot$, where x and y are functions of the zenith-distances. By way of example we have for Munich and $z=75°$ (under the supposition that the corrections hold good for the middle of the month) the following formula : Correction $=-0''\!\cdot\!03-0''\!\cdot\!34\sin\odot+0''\!\cdot\!21\cos\odot$. Supposing, then, the influence of moisture to have been omitted in the computation of refraction, the declinations (derived from the zenith-distances) of stars observed south of the zenith and the declinations concluded from the observations of northern stars at their lower culminations require a positive correction if observed in winter and a negative correction if observed in summer; and these statements must be reversed for stars in upper culmination observed north of the zenith.

Besides the influence of damp here considered, there is still another effect which indeed almost disappears at $z=75°$, but is noticeable at greater zenith-distances, and must be alluded to in this reference to M. Radau's memoir, *Essai sur les Réfractions Astronomiques*, pp. 16, 17. On the other hand we must not fail to mention that M. Radau on the strength of the experiments of

Fizeau and Jamin, comes to the conclusion that the equation (7) should be replaced by

$$(7a) \qquad \rho = p \frac{1 - \frac{1}{8}\frac{\pi}{p}}{1 + at}$$

At the present time there are only two papers * published, in which, by means of astronomical observations, an attempt has been made to decide the question whether the formula (7) or (7a) is to be employed for the density of the air. Even if both investigations lead to the result that the former formula is to be preferred, yet a renewed examination of the question on the basis of the materials collected at other observatories is very desirable. In the meantime the corrections calculated by the writer, adopting formula (7), may be of interest.

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On the Formula connecting Diameters of Photographic Images with Stellar Magnitude. By H. H. Turner, D.Sc., F.R.S., Savilian Professor.

1. The measures of the Oxford portion of the Astrographic Catalogue being now completed, it is possible to formulate certain conclusions as to the behaviour of the different plates in portraying stellar images. Early in the history of the work it was decided to record the diameters of the stellar images as an indication of the magnitude, and this was done throughout. But discussion of the results was deferred until a large amount of material had been accumulated.

2. The diameter of each star disc was estimated in units of 0″·3 in both positions of the plate, and the mean of the two estimates has been set down. No great precision is claimed for these estimates, and they are affected by a number of circumstances such as the following:

(*a*) Personal habits of the different measurers in estimating the limits of the ill-defined disc.

(*b*) Elongation of images near the corners of the plate. The observer was instructed to take the mean of the two diameters.

(*c*) Differences in intensity of image for the faint stars. In place of an estimate of diameter we have, then, an estimate of faintness.

3. In addition to these difficulties of interpreting the actual record on the plate, there is a systematic change in the impression

* Bauschinger, "Untersuchungen über die astronomische Refraktion," *Annals of the Munich Observatory*, vol. iii.; Courvoisier, "Unters. üb. die astr. Refr.," *Publications of the Heidelberg Observatory*, vol. iii.